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LETTER TO THE EDITOR

Reciprocal transformations in (2+1) dimensions

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Abstract. Reciprocal Bäcklund transformations are introduced for non-linear integro-differential equations in (2+1) dimensions. Invariance under such transformations is investigated.

The subject of reciprocal transformations and their application in gas dynamics and magneto(gas dynamics) has been surveyed in Rogers and Shadwick (1982). The notion of reciprocal Bäcklund transformations of conservation laws was introduced by Kingston and Rogers (1982, 1984). Such transformations have been used to solve important non-linear boundary value problems (Rogers *et al* 1983, Rogers 1983, 1985a). In particular, they have recently been used to solve a class of Stefan problems in non-linear heat conduction (Rogers 1985b). In another context, the notion of reciprocally related inverse scattering schemes was recently introduced and shown to be a key component in the link between the AKNS and WKI schemes (Rogers and Wong 1984).

Here, the concept of reciprocal transformations is extended to (2+1) dimensions and an invariance result is established.

Integro-differential equations of the form

$$(\partial/\partial t)T(\partial_x;\partial_y;\partial_x^{-1}\colon u(x,y,t))+\Phi(\partial_x;\partial_y;\partial_x^{-1}\colon u(x,y,t))=0 \tag{1}$$

are considered where $\partial_x := (\partial/\partial x)$, $\partial_y := (\partial/\partial y)$ and ∂_x^{-1} is defined by

$$\partial_x^{-1}\phi := \int_a^x \phi(\sigma, y, t)\,\mathrm{d}\sigma. \tag{2}$$

Reciprocal transformations of the type

$$\left.\begin{array}{lll} x'=\partial_x^{-1}T & y'=y & t'=t \\ 0<|T|<\infty & & \end{array}\right\}R \tag{3}$$

are introduced, where it is required that the involutory condition $R^2=I$ hold. If Φ admits a potential

$$F=-\partial_x^{-1}T_t \tag{4}$$

(1) becomes

$$\frac{\partial}{\partial t}T(\partial_x;\partial_y;\partial_x^{-1}\colon u(x,y,t))+\frac{\partial}{\partial x}F(\partial_x;\partial_y;\partial_x^{-1}\colon u(x,y,t))=0 \tag{5}$$

while (3) yields

$$\mathrm{d}x' = T\mathrm{d}x - F\mathrm{d}t + \partial_x^{-1} T_y \mathrm{d}y \qquad y' = y \qquad t' = t$$
$$0 < |T| < \infty. \tag{6}$$

Accordingly, if we set

$$H = \partial_x^{-1} T_y \tag{7}$$

then the condition $\mathrm{d}x'' = \mathrm{d}x$ yields

$$\mathrm{d}x'' = T'\mathrm{d}x' - F'\mathrm{d}t' + H'\mathrm{d}y'$$
$$= T'\{T\mathrm{d}x - F\mathrm{d}t + H\mathrm{d}y\} - F'\mathrm{d}t + H'\mathrm{d}y = \mathrm{d}x$$

whence we obtain the reciprocal relations

$$T' = 1/T \tag{8}$$

$$F' = -F/T \tag{9}$$

$$H' = -H/T. \tag{10}$$

Moreover, under R it is seen that

$$\partial_x = T\partial_{x'} \tag{11}$$

$$\partial_y = (\partial_x^{-1} T_y)\partial_{x'} + \partial_{y'} \tag{12}$$

$$\partial_t = -F\partial_{x'} + \partial_{t'} \tag{13}$$

together with

$$\partial_x^{-1} = \partial_{x'}^{-1} \boldsymbol{T}' \tag{14}$$

where $\boldsymbol{T}'$ is the multiplicative operator defined by

$$\boldsymbol{T}'\phi := T'\phi. \tag{15}$$

The reciprocal operator relation (14) is readily established. Thus let

$$v = \partial_x^{-1}\phi \tag{16}$$

$$w = \partial_{x'}^{-1}\phi \tag{17}$$

so that

$$\partial_x v = \partial_{x'} w = \phi. \tag{18}$$

But

$$\partial_{x'} = T'\partial_x \tag{19}$$

so that

$$\phi = \partial_{x'} w = \frac{1}{T'}\partial_{x'} v = \frac{1}{T'}\partial_{x'}[\partial_x^{-1}\phi]$$

whence

$$\boldsymbol{T}'\phi = \partial_{x'}[\partial_x^{-1}\phi]$$

or

$$\partial_{x'}^{-1}\boldsymbol{T}'\phi = \partial_x^{-1}\phi. \tag{20}$$

The result (14) follows, as does its reciprocal analogue

$$\partial_{x'}^{-1} = \partial_x^{-1} \boldsymbol{T} \tag{21}$$

$$\boldsymbol{T}\phi := T\phi. \tag{22}$$

Use of the latter shows that, as required,

$$\begin{aligned}\partial_{x'}^{-1} T'_{y'} &= \partial_x^{-1}[TT'_{y'}] \\ &= \partial_x^{-1}\left(\frac{-1}{T}\{T_y - HT_{x'}\}\right) = \partial_x^{-1}\left(\frac{-T_y}{T} + \frac{H}{T^2}T_x\right) = \frac{-H}{T} = H'\end{aligned} \tag{23}$$

on application of the relation $H_x = T_y$.

Accordingly, the following reciprocal result has been established.

Theorem. The integro-differential equation

$$\partial_t T(\partial_x\,;\partial_y\,;\partial_x^{-1}\!: u(x,y,t)) + \partial_x F(\partial_x\,;\partial_y\,;\partial_x^{-1}\!: u(x,y,t)) = 0 \tag{24}$$

is transformed under the reciprocal transformation R to the associated equation

$$\partial_{t'} T' + \partial_{x'} F' = 0 \tag{25}$$

where

$$T' = \frac{1}{T(D_1';\, D_2';\, D_3';\, u)} \tag{26}$$

$$F' = \frac{-F(D_1';\, D_2';\, D_3';\, u)}{T(D_1';\, D_2';\, D_3'\!:\, u)} \tag{27}$$

$$D_1' := \partial_x = \frac{1}{T'}\partial_{x'} \tag{28}$$

$$D_2' := \partial_y = \frac{-(\partial_{x'}^{-1} T'_{y'})}{T}\partial_{x'} + \partial_{y'} \tag{29}$$

$$D_3' := \partial_x^{-1} = \partial_{x'}^{-1}\boldsymbol{T}'. \tag{30}$$

The above represents a generalisation to (2+1) dimensions of the reciprocal result due to Kingston and Rogers (1982).

Suppose that

$$\frac{\partial T_1}{\partial t_1} + \frac{\partial F_1}{\partial x_1} = 0 \tag{31}$$

where

$$F_1 = F_1(\partial_{x_1}\,;\partial_{y_1}\,;\partial_{x_1}^{-1}\!: T_1) \tag{32}$$

is invariant under the involutory transformation

$$\tau\!: T_1 \to \bar{T}_1 = \frac{1}{T_1}. \tag{33}$$

Thus, under τ, (31) becomes

$$\frac{\partial \bar{T}_1}{\partial t_1} + \frac{\partial \bar{F}_1}{\partial x_1} = 0 \tag{34}$$

where

$$\bar{T}_1 = 1/T_1 \tag{35}$$

$$\bar{F}_1 = F_1\left(\partial_{x_1};\partial_{y_1};\partial_{x_1}^{-1}:\frac{1}{\bar{T}_1}\right). \tag{36}$$

Application of the reciprocal transformations

$$\left.\begin{array}{lll} x_2 = \partial_{x_1}^{-1} T_1 & y_2 = y_1 & t_2 = t_1 \\ 0 < |T_1| < \infty & & \end{array}\right\} R_1 \tag{37}$$

and

$$\left.\begin{array}{lll} \bar{x}_2 = \partial_{x_1}^{-1} \bar{T}_1 & \bar{y}_2 = y_1 & \bar{t}_2 = t_1 \\ 0 < |\bar{T}_1| < \infty & & \end{array}\right\} R_2 \tag{38}$$

to (31) and (34) in turn produces the reciprocal equations

$$\frac{\partial T_2}{\partial t_2} + \frac{\partial F_2}{\partial x_2} = 0 \tag{39}$$

and

$$\frac{\partial \bar{T}_2}{\partial \bar{t}_2} + \frac{\partial \bar{F}_2}{\partial \bar{x}_2} = 0 \tag{40}$$

where

$$\begin{array}{ll} T_2 = \dfrac{1}{T_1} & F_2 = -\dfrac{F_1}{T_1} \\ \bar{T}_2 = \dfrac{1}{\bar{T}_1} & \bar{F}_2 = -\dfrac{\bar{F}_1}{\bar{T}_1}. \end{array} \tag{41}$$

Now, (37) and (38) together show that

$$\left.\begin{array}{lll} \mathrm{d}\bar{x}_2 = T_3 \mathrm{d}x_2 - F_3 \mathrm{d}t_2 + H_3 \mathrm{d}y_2 & \mathrm{d}\bar{y}_2 = \mathrm{d}y_2 & \mathrm{d}\bar{t}_2 = \mathrm{d}t_2 \\ 0 < |T_3| < \infty & & \end{array}\right\} R \tag{42}$$

where

$$\begin{array}{lll} T_3 = \dfrac{\bar{T}_1}{T_1} & F_3 = \bar{F}_1 - \dfrac{\bar{T}_1 F_1}{T_1} & H_3 = \bar{H}_1 - \dfrac{\bar{T}_1 H_1}{T_1} \\ H_1 = \partial_{x_1}^{-1} T_{1,y_1} & \bar{H}_1 = \partial_{x_1}^{-1} \bar{T}_{1,y_1}. & \end{array} \tag{43}$$

Application of the reciprocal transformation R to

$$\frac{\partial T_3}{\partial t_2} + \frac{\partial F_3}{\partial x_2} = 0 \tag{44}$$

produces

$$\frac{\partial T_3^*}{\partial \bar{t}_2} + \frac{\partial F_3^*}{\partial \bar{x}_2} = 0 \tag{45}$$

where

$$T_3^* = \frac{1}{T_3} = \frac{T_1}{\bar{T}_1} \tag{46}$$

$$F_3^* = \frac{-F_3}{T_3} = F_1 - \frac{T_1 \bar{F}_1}{\bar{T}_1}. \tag{47}$$

Moreover, since τ is an invariant transformation, the symmetry of the above procedure shows that R must be a reciprocal *auto-Bäcklund transformation* between (44) and (45). The construction is illustrated in figure 1.

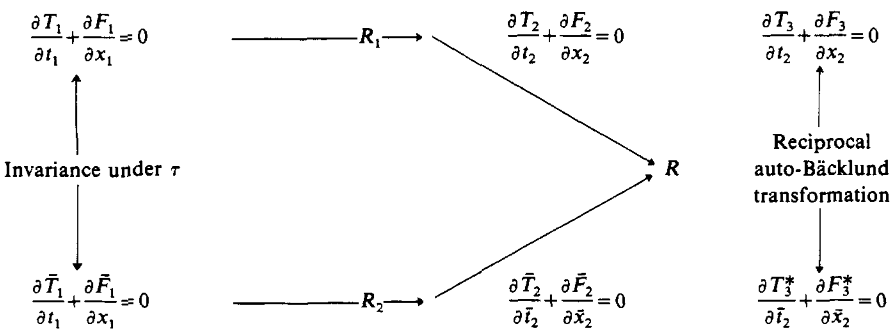

Figure 1. Generation of reciprocal auto-Bäcklund transformations.

Now, (35) and (36) together with (43) show that

$$T_3 = T_1^{-2} \tag{48}$$

$$F_3 = F_1\left(\partial_{x_1}; \partial_{y_1}; \partial_{x_1}^{-1}: \frac{1}{T_1}\right) - \frac{1}{T_1^2} F_1(\partial_{x_1}: \partial_{y_1}; \partial_{x_1}^{-1}: T_1)$$

$$= \frac{1}{T_1}[J(\partial_{x_1}; \partial_{y_1}; \partial_{x_1}^{-1}: 2 \ln T_1) - J(\partial_{x_1}; \partial_{y_1}; \partial_{x_1}^{-1}: -2 \ln T_1)] \tag{49}$$

where

$$J(\partial_{x_1}; \partial_{y_1}; \partial_{x_1}^{-1}: \alpha) := -e^{-\alpha/2} F_1(\partial_{x_1}; \partial_{y_1}; \partial_{x_1}^{-1}: e^{\alpha/2}). \tag{50}$$

In terms of T_3, x_2 and y_2, (49) yields

$$F_3 = T_3^{1/2}[J(T_3^{-1/2}\partial_{x_2}; -T_3^{-1/2}(\partial_{x_2}^{-1} T_{3,y_2}^{1/2})\partial_{x_2} + \partial_{y_2}; \partial_{x_2}^{-1} T_3^{1/2}: -\ln T_3)$$
$$- J(T_3^{-1/2}\partial_{x_2}; -T_3^{-1/2}(\partial_{x_2}^{-1} T_{3,y_2}^{1/2})\partial_{x_2} + \partial_{y_2}; \partial_{x_2}^{-1} T_3^{1/2}; \ln T_3)]. \tag{51}$$

Accordingly, it is seen that the class of non-linear integro-differential equations (44) where F_3 adopts the form (51) is *invariant* under the reciprocal transformation

$$\bar{x}_2 = \partial_{x_2}^{-1} T_3 \qquad \bar{y}_2 = y_2 \qquad \bar{t}_2 = t_2$$
$$T_3^* = 1/T_3.$$

The above result represents an extension to $(2+1)$ dimensions of the reciprocal invariance property established by Kingston *et al* (1984).

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